Color Superconductivity

Condensed matter physics of quark matter

Contents

1. Introduction
2. Color superconductivity at an ultra-high baryon density ----------- First principle calculation of QCD
3. Color superconductivity at a realistic baryon density ------------ NJL effective action approach
4. Concluding remarks

I. Introduction

- *Quarks of different color and flavors*:

	up	down	strange	charm	bottom	top
	u	*d*	*s*	*c*	*b*	*t*
	u	*d*	*s*	*c*	*b*	*t*
	u	*d*	*s*	*c*	*b*	*t*
Mass	1.5-4MeV	4-8MeV	100-200MeV	1.3GeV	5GeV	170GeV
Charge	2/3	-1/3	-1/3	2/3	-1/3	2/3
Baryon no.	1/3	1/3	1/3	1/3	1/3	1/3

- *Quark contents of hadrons*:

  p = (uud) *n* = (udd), etc.

The relevant quark flavors for color superconductivity are u, d & s.

- ***Dynamics of quarks ---- QCD* (quantum chromodynamics)**

Yang-Mills theory of SU(3) color gauge group:

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}F_{\mu\nu}F_{\mu\nu} - \bar{\psi}\gamma_{\mu}\left(\frac{\partial}{\partial x_{\mu}} - igA_{\mu}\right)\psi + \bar{\psi}m\psi$$

$$+ \text{gauge fixing terms} + \text{renormalization counterterms}$$

where

$$F_{\mu\nu} = \frac{\partial A_{\nu}}{\partial x_{\mu}} - \frac{\partial A_{\mu}}{\partial x_{\nu}} - ig\left[A_{\mu}, A_{\nu}\right]$$

$A_{\mu} = \frac{1}{2}A_{\mu}^{l}\lambda_{l}$ stands for gluon field with $\mu = 1,2,3,4$ and $l = 1,\ldots,8$

ψ denotes the quark 4-spinor. It carries three color indexes and N_f flavor indexes. For CSC $N_f = 2 \sim 3$

m is the mass matrix in the flavor space

λ's are 3x3 Gell-Mann matrices given by

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Quark field ψ supports the fundamental representation, **3** of the color SU(3)

Gluon field A_μ^l supports the adjoint representation, **8.**

- ***Properties of QCD***

Asymptotic freedom (anti-screening):

QCD running coupling $$\alpha_s = \frac{g^2}{4\pi} = \frac{2\pi}{9\ln\frac{E}{\Lambda}}$$

$$E = \max(T, \mu) \quad \text{for } T >> \Lambda, \text{or } \mu >> \Lambda$$

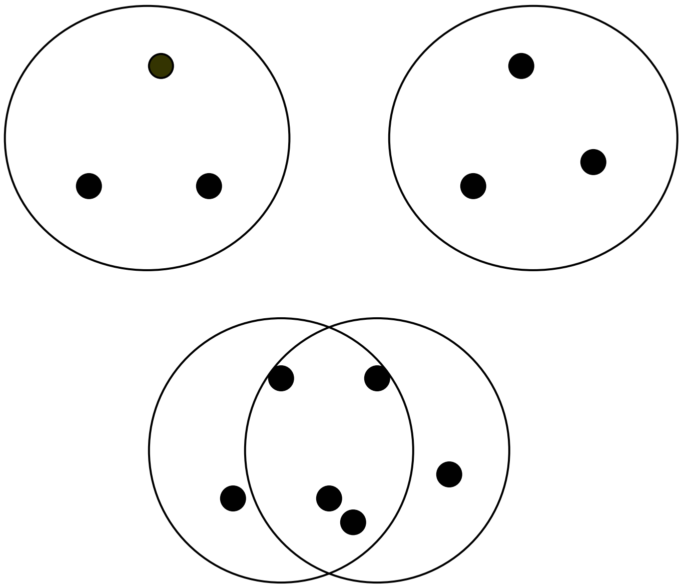
$$T = \text{temperature}$$

$$\mu = \text{chemcal potential}$$

$$\Lambda \cong 200\text{MeV}$$

Baryon density	Fermi energy(MeV)	α_S
$10n_B^{(0)}$	442	0.88
$1000000n_B^{(0)}$	20550	0.151

- Quark confinement --------Elementary excitations of QCD at low temperature and chemical potential are hadrons (nucleons and mesons).
- Quark deconfinement ------Elementary excitations at high temperature or high chemical potential are quarks and gluons.

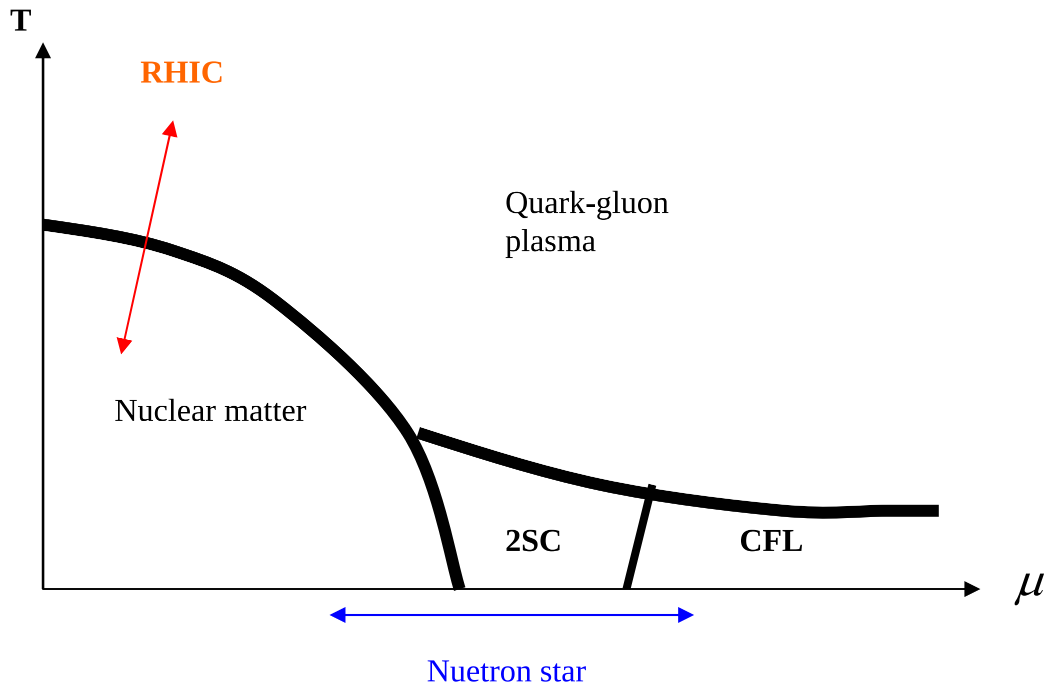


MIT bag model predicts the crossover baryon density from nuclear matter to quark matter

$$n_B \sim 10 n_B^{(0)}$$

nucleons loses identities under squeezing

QCD phase diagram



RHIC: relativistic heavy ion collision
2SC & CFL ---------- two color superconducting phases

Experimental tools:

- RHIC: high temperature, low chemical potential;
- Neutron star observation: low temperature high chemical potential.

Theoretical tools:

- Perturbation theory: high temperature or high chemical potential;
- Lattice simulation: intermediate temperature and low chemical potential;
- NJL effective action: low temperature and intermediate chemical potential;
- AdS/CFT correspondence: decomfined phase.

II. Color superconductivity at an ultra-high baryon density

- *Zero-th order*

Chemical potential $\mu >> \Lambda$ such that $\alpha_S << 1$

A fermi sea of u, d, s quarks of equal Fermi momenta

The whole system is electrically neutral and is ulta-relativistic.

Kinetic energy relative to the Fermi energy (=chemical potential)

$$\xi_p = \sqrt{p^2 + m^2} - \mu \cong p - \mu$$

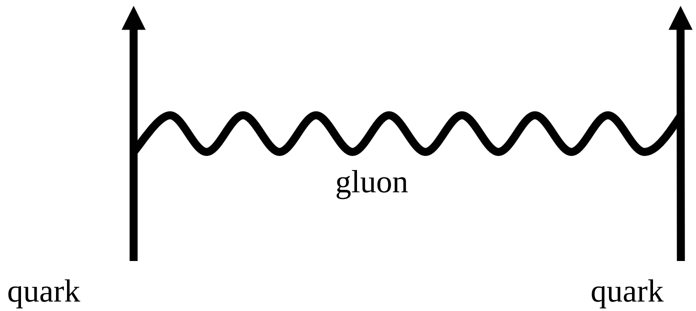
Quark number density for each color and flavor:

$$n_{c,f} = \frac{k_F^3}{3\pi^2}$$

Baryon number density = (1/3) X quark number density

- *Pairing via one-gluon-exchange*

Barrois; Frautschi; Bailin & Love



$$\propto g^2 \lambda_l^{c_1 c_1'} \lambda_l^{c_2 c_2'}$$

$$(\lambda_l)^{c_1 c_1'} (\lambda_l)^{c_2 c_2'} = -\frac{4}{3}\left(\delta^{c_1' c_1}\delta^{c_2' c_2} - \delta^{c_1' c_2}\delta^{c_2' c_1}\right) + \frac{2}{3}\left(\delta^{c_1' c_1}\delta^{c_2' c_2} + \delta^{c_1' c_2}\delta^{c_2' c_1}\right)$$

$$\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$$

color antisymmetric
attractive

color symmetric
repulsive

Favorite pairing channel: different colors, different flavors, s-wave

- *Temperature of Pairing Instability*

Son; Schaefer & Wilczek; Hong, Miransky &Shovkovy; Brown, Liu & Ren; Wang & Rischke; Manuel

$$T_p = \frac{2048}{3}\sqrt{\frac{2}{3}}\frac{\mu}{g^5} e^{-\frac{3\pi^2}{\sqrt{2}g}+\gamma_E-\frac{\pi^2+4}{8}}\left[1+O(g)\right]$$

---- Non BCS scaling: $\ln T_p \sim \frac{1}{g}$ instead of $\ln T_p \sim \frac{1}{g^2}$

---- Strong type I.

- *Transition Temperature*

$$T_c = \left(1+\frac{\pi^2}{12\sqrt{2}}g\right)T_p$$

Giannakis, Hou, Ren & Rischke

---- Gauge filed fluctuation renders the phase transition first order, the Mechanism of Halperin, Lubinsky and Ma.

- *Energy gap at zero temperature*

Schaefer & Wilczek; Pisarski & Rischke; Reuter; Feng, Hou, Li & Ren

---- Solution to the Eliashberg equation:

$$\Delta(\omega) = \begin{cases} \Delta_0 & \text{for } 0 < \omega < \Delta_0 \\ \Delta_0 \cos\left[\frac{g}{3\sqrt{2\pi}}\left(\ln\frac{|\omega|}{\Delta_0} - i\frac{\pi}{2}\right)\right] & \text{for } \omega > \Delta_0 \end{cases}$$

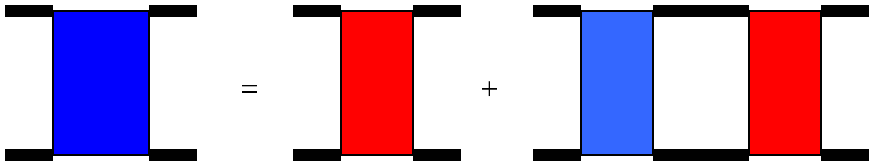
$$\Delta_0 = 2^{-\frac{1}{3}} \pi e^{-\gamma_E} T_p$$

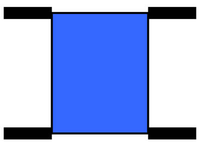
---- quasi particle pole:

$$\omega^2 - (p-\mu)^2 - \Delta^2(\omega) = 0$$

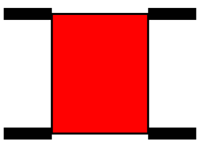
• *Calculation details*

Schwinger-Dyson equation:





= proper vertex of diquark scattering = $\Gamma(P' \mid P)$



= 2PI vertex of diquark scattering = $\gamma(P' \mid P)$

= dressed quark propagator = S(Q)

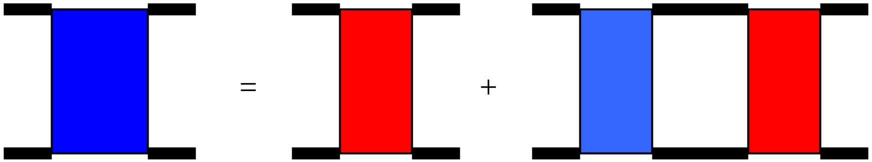
Incoming quark momenta - energies $= P \equiv (\pm\mathbf{p}, \pm i\,\nu)$

Outgoing quark momenta - energies $= P' \equiv (\pm\mathbf{p}', \pm i\,\nu')$

ν and ν' = Matsubara energies

- *Calculation details*

Schwinger-Dyson equation:



$$\Gamma\left(P' \mid P\right) = \gamma(P' \mid P) + \frac{T}{\Omega}\sum_{Q}\gamma(P' \mid Q)S(Q)S(-Q)\Gamma(Q \mid P)$$

$$= \gamma(P' \mid P) + \sum_{Q} K(P' \mid Q)\Gamma(Q \mid P)$$

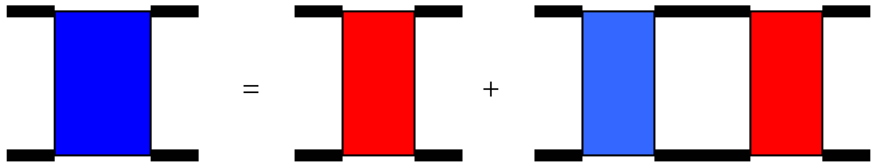
Fredholm integral equation for $\Gamma(P' \mid P)$

$D(T,\mu) \equiv$ the Fredholm determinant

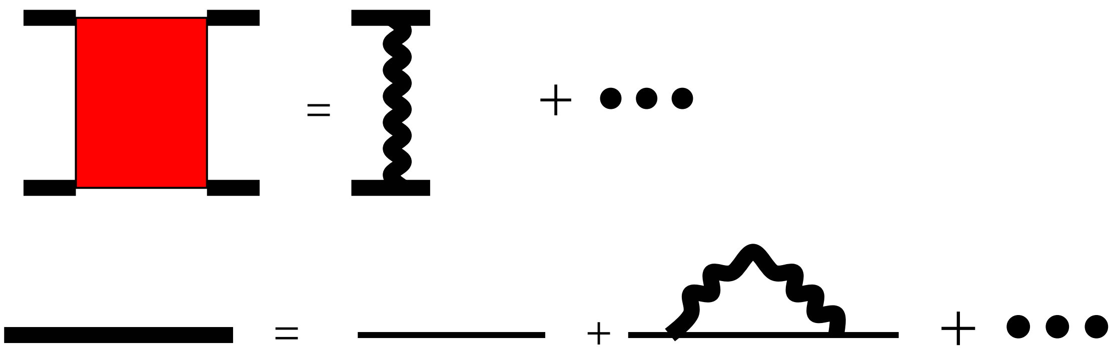
The temperature of pairing instability
= the highest temperature such that $D(T,\mu) = 0$

- *Calculation details*

Schwinger-Dyson equation:



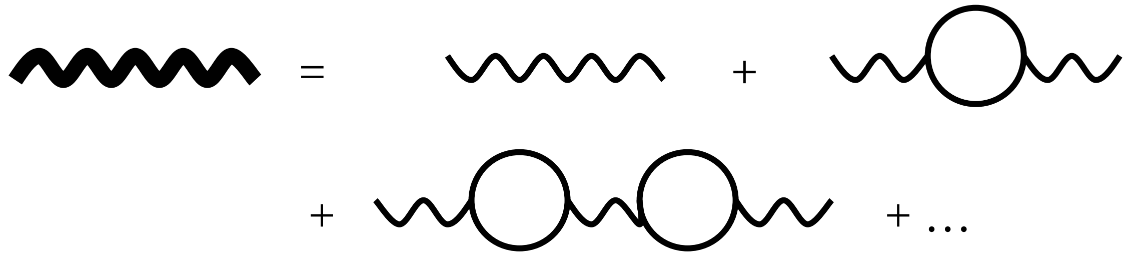
Perturbative expansion:



 = hard dense loop (HDL) resummed gluon propagator

HDL gluon propagator:

$$D_{\mu\nu}(K) =$$



$$= \begin{cases} \dfrac{-1}{k^2 + k_0^2 + \sigma_M(k,k_0)}\left(\delta_{ij} - \dfrac{k_i k_j}{k^2}\right) & \text{for } \mu,\nu = i, j \\ \dfrac{-1}{k^2 + \sigma_E(k,k_0)} & \text{for } \mu = \nu = 0 \end{cases}$$

For pairing, $k_0 << k << \mu$

Color magnetic self energy $\sigma_M(k,k_0) \cong \dfrac{\pi}{4l_D^2}\dfrac{k_0}{k}$ Landau damping

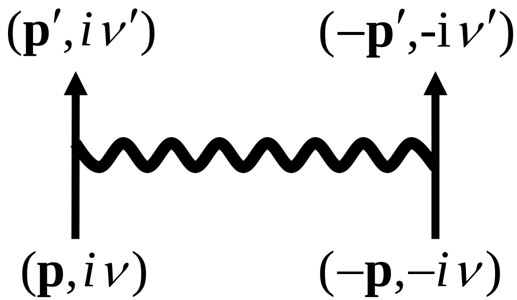
Color electric self energy $\sigma_E(k,k_0) = \dfrac{1}{l_D^2}$

l_D is the Debye length $\dfrac{1}{l_D^2} = \dfrac{3g^2\mu^2}{2\pi^2}$

Magnetic gluon dominate the pairing

Forward singularity, non BCS scaling

Pairing potential, $V \sim \int_{-1}^{1} d\cos\theta \Big($



$\Big) \sim g^2 \ln \frac{1}{|\nu - \nu'|}$

where $\cos\theta \equiv \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$

Pairing instability:

$$1 \sim T \sum_{\nu} V \int_{-\infty}^{\infty} d(p-\mu) \frac{1}{(p-\mu)^2 + \nu^2} \sim g^2 T \sum_{\nu} \left(\ln \frac{1}{|\nu|} \right) \frac{1}{|\nu|}$$

$$\sim g^2 \int_{T_p}^{\mu} \frac{d\nu}{\nu} \ln \frac{1}{\nu} \sim g^2 \ln^2 \frac{1}{T_p}$$

Careful analysis yields $\quad T_p = b \frac{\mu}{g^5} \exp\left(-\frac{3\pi^2}{\sqrt{2}g} \right)$ ***Son***

Collecting constant pertaining to the logarithm

$$b = \frac{2048}{3} \sqrt{\frac{2}{3}} e^{\gamma_E} b'$$

Schaerfer & Wilczek; Pisarski & Rischke

Non Fermi liquid behavior

Quark self energy

$$\Sigma(P) = -i\frac{g^2}{9\pi^2}\gamma_4\nu\ln\frac{k_c^3}{m_D^2|\nu|} = O(g) \qquad \text{for } \nu \sim T_p$$

$$\text{where } k_c >> T_p$$

---- Generalization of the QED results of Holstein, Norton, Picus and Reizer

Contribution to the pairing temperature: ***Brown, Liu & Ren***

$$T_p = \frac{2048}{3}\sqrt{\frac{2}{3}}\frac{b''\mu}{g^5}\exp\left(-\frac{3\pi^2}{\sqrt{2}g} + \gamma_E - \frac{\pi^2+4}{8}\right)$$

Careful examination of other higher order diagrams leads to

$$b'' = 1 + O(g)$$ ***Brown, Liu & Ren***

Done!

- *Color-flavor locking (CFL) below the transition temperature*

Alford, Rajagopal & Wilczek

---- Energetically favored condensate

$$< \psi_{f_1}^{c_1} \psi_{f_2}^{c_2} >= \phi\left(\delta_{f_1}^{c_1}\delta_{f_2}^{c_2} - \delta_{f_2}^{c_1}\delta_{f_1}^{c_2}\right)$$

----Symmetry breaking

$$SU(3)_c \otimes SU(3)_{f_R} \otimes SU(3)_{f_L} \otimes U(1)_B \rightarrow SU(3)_{c+f_R+f_L} \otimes Z_2$$

----Analogous to the AB phase of the superfluid of $^3\mathrm{He}$

where orbital angular momentum is locked with spin, the symmetry breaking is

$$\text{(orbital rotation)} \otimes \text{(spin rotation)} \rightarrow \text{(simultaneous rotation)}$$

III. Color superconductivity at a realistic baryon density

- *The density accessible inside the core of a neutron star*

$$\mu = 400 \sim 500\mathrm{MeV} \text{ and } \alpha_S \cong 0.76 \sim 1.00$$

------ One-gluon-exchange gives transition temperature around 3.7MeV (=43,000,000,000K) at 500MeV, but the perturbative result may not be reliable;

------ Nonperturbative effects may end up with higher transition temperature;

------ The mass of strange quark cannot be neglected;

------ Crossover from 3 flavor (u, d, s) color superconductivity to 2 flavor (u, d) color superconductor

------ Fermi momentum mismatch between different flavors suggest a number of exotic superconductivity, e.g. Sarma state, LOFF, phase separation.

- *Fermi momentum mismatch*

Consider an ideal gas of three quark favors and electrons at $T = 0$

The energy density

$$E = \frac{3}{\pi^2}\int_0^{k_u} dp p^2 \sqrt{p^2 + m_u^2} + \frac{3}{\pi^2}\int_0^{k_d} dp p^2 \sqrt{p^2 + m_d^2} + \frac{3}{\pi^2}\int_0^{k_s} dp p^2 \sqrt{p^2 + m_s^2}$$

$$+ \frac{1}{\pi^2}\int_0^{k_u} dp p^2 \sqrt{p^2 + m_e^2}$$

The baryon number density $n_B = \frac{1}{3}(n_u + n_d + n_s) \equiv \frac{k_0^3}{\pi^2}$

The electric charge density $n_Q = \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e = 0$

where $n_f = \frac{k_f^2}{\pi^2} \quad f = u, d, s \qquad n_e = \frac{k_e^2}{3\pi^2}$

Charge neutrality: $n_Q = 0$

Pressure $p = \mu_b b + \mu_q q - E$

Equilibrium condition:

$$\left(\frac{\partial p}{\partial k_u}\right)_{\mu_b, \mu_q} = \left(\frac{\partial p}{\partial k_d}\right)_{\mu_b, \mu_q} = \left(\frac{\partial p}{\partial k_s}\right)_{\mu_b, \mu_q} = \left(\frac{\partial p}{\partial k_e}\right)_{\mu_b, \mu_q} = 0$$

- *Fermi momentum mismatch*

The solution at $k_0 >> m_s$ $k_0 = \left(\pi^2 n_B\right)^{\frac{1}{3}}$

$$k_u \cong k_0$$

$$k_d \cong k_0 + \frac{m_s^2}{4k_0}$$

$$k_s \cong k_0 - \frac{m_s^2}{4k_0}$$

$$k_e \cong \frac{m_s^2}{4k_0}$$

3 flavor color superconductivity

The solution at $m_u, m_d << k_0 << m_s$

$$k_u \cong 0.87 k_0$$

$$k_d \cong 1.09 k_0$$

$$k_e \cong 0.22 k_0$$

2 flavor color superconductivity (2SC)

- *Two flavor NJL(Nambu-Jona-Lasinio) effective Lagrange* ***Rapp, Schaefer, Shuryak &Velkovsky***

$$\mathrm{L} = -\bar{\psi}\gamma_\mu \frac{\partial \psi}{\partial x_\mu} + G_S\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}\vec{\tau}\psi\right)^2\right] + G_D\left(\bar{\psi}_C\gamma_5\varepsilon^c\tau_2\psi\right)\left(\bar{\psi}_C\gamma_5\varepsilon^c\tau_2\psi\right)$$

where $\psi_C = \gamma_2\psi^*$, the three 3x3 matrices $\left(\varepsilon^c\right)^{ab} = \varepsilon^{cab}$

act on color indexes (red, green and blue) and the second Pauli matrix τ_2

acts on flavor indexes (u, d).

$G_S \cong 5.32\,\mathrm{GeV}^{-2}, G_D \cong 4\,\mathrm{GeV}^{-2}$ and UV cutoff $\Lambda_f \cong 653\,\mathrm{MeV}$.

(estimated by fitting the hadron masses at zero chemical potential.)

---- Effective four fermion point interaction inspired by QCD instantons;
---- The last term (diquark coupling) gives rise to CSC;
---- Pairing channel is color antisymmetric, like the one-gluon exchange;
---- s-wave pairing between different colors and different flavors, e.g. (u, d) and (u, **d**);
---- Mean field theory solution;
---- Share some features of a cold atom system.

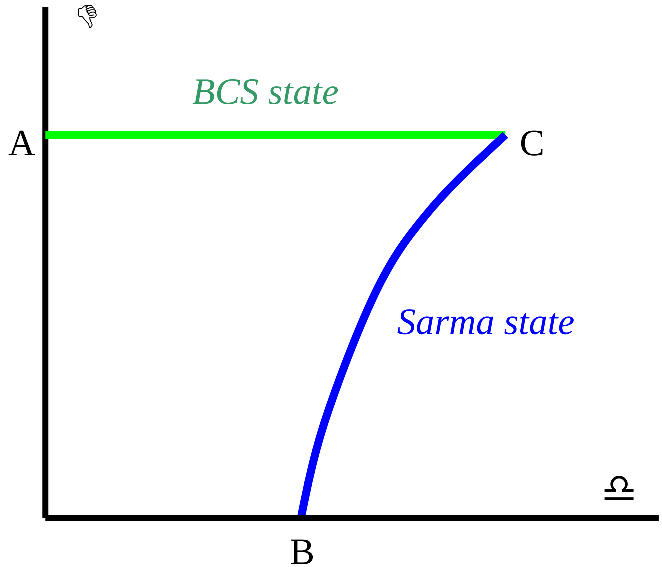
- *Exotic superconductivity under mismatch*

---- Pairing between Fermi Momenta ❍🖅♎ and ❍📪♎🖃

---- Point attraction (BCS like).

$$\mu = \frac{1}{3}\mu_B + \frac{1}{6}\mu_Q, \quad \delta = -\frac{1}{3}\mu_Q$$

Homogeneous condensate $< \psi\psi > = \text{constant}$



$$A = (0, \Delta_0)$$

$$B = \left(\frac{\Delta_0}{2}, 0 \right)$$

$$C \cong \left(\Delta_0, \Delta_0 \right)$$

$$\Delta_0 \equiv \text{the solution at } T = 0.$$

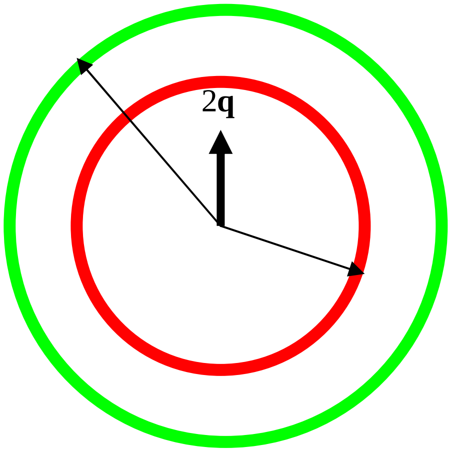
Solution to the gap equation

LOFF (Larkin-Ovchinnikov-Fulde-Ferrell) condensate:

---- Single plane wave

$$<\psi\psi> \propto e^{2i\mathbf{q}\cdot\mathbf{r}}$$

Choose q to minimize the thermodynamic potential
Energetically more favored than the normal, BCS and Sarma phases within the LOFF window



$$0.706\Delta_0 < \delta < 0.754\Delta_0 \qquad q \cong 1.2\Delta_0$$

---- Multi-plane wave

$$<\psi\psi> \propto \sum_{\mathbf{q}} \Delta_{\mathbf{q}} e^{2i\mathbf{q}\cdot\mathbf{r}}$$

May lower the thermodynamic potential further, difficult to treat.

- *2SC-BCS*

$$\mu = \frac{1}{3}\mu_B + \frac{1}{6}\mu_Q, \quad \delta = -\frac{1}{3}\mu_Q$$

---- Excitation spectrum

$$\varepsilon_p^{\pm} = \sqrt{(p-\mu)^2 + \Delta^2} \pm \delta \quad \Delta > \delta$$

Gapped

---- Thermodynamic potential

$$\Gamma \equiv -\frac{T}{\Omega}\ln\exp\left(-\frac{H - \mu_B B - \mu_Q Q}{T}\right) = -p$$

$$\cong \Gamma_n(\mu,\delta) + \frac{2\mu^2}{\pi^2}\left[\delta^2 - \Delta^2\left(\ln\frac{2\omega_0}{\Delta} + \frac{1}{2}\right)\right] + \frac{\Delta^2}{4G_D}$$

where

$$\Gamma_n(\mu,\delta) = -\frac{1}{4\pi^2}\left[(\mu-\delta)^4 + (\mu+\delta)^4\right]$$

---- Equilibrium condition

$$\left(\frac{\partial\Gamma}{\partial\Delta}\right)_{\mu_B,\mu_Q} = 0 \quad \Rightarrow \quad \Delta \cong 2\omega_0 e^{-\frac{\pi^2}{8G_D\mu^2}} \equiv \Delta_0$$

---- At the equilibrium

$$\Gamma \cong \Gamma_n(\mu,\delta) + \frac{\mu^2}{\pi^2}\left(2\delta^2 - \Delta_0^2\right) \qquad \Gamma > \Gamma_n \qquad \text{for } \delta > \frac{1}{\sqrt{2}}\Delta_0$$

---- Charge neutrality may be restored by unpaired quarks

- *g2SC* (Sarma state)

Liu & Wilczek,
Huang, Zhuang & Zhao,
Shovkovy & Huang

---- Excitation spectrum

$$\varepsilon_p^{\pm} = \left|\sqrt{(p-\mu)^2 + \Delta^2} \pm \delta\right| \quad \Delta < \delta$$

Gapless

---- Thermodynamic potential

$$\Gamma \cong \Gamma_n(\mu,\delta) + \frac{2\mu^2}{\pi^2}\left(\Delta^2 \ln\frac{\delta+\sqrt{\delta^2-\Delta^2}}{\Delta_0} - \delta\sqrt{\delta^2-\Delta^2} - \frac{1}{2}\Delta^2 + \delta^2\right)$$

---- Equilibrium condition

$$\left(\frac{\partial\Gamma}{\partial\Delta}\right)_{\mu_B,\mu_Q} = 0 \qquad \Rightarrow \qquad \Delta = \sqrt{\Delta_0\left(2\delta - \Delta_0\right)}$$

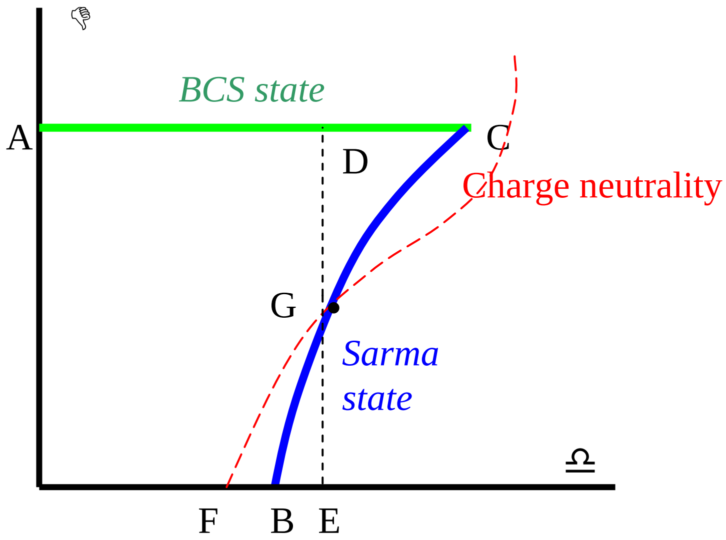
---- At the equilibrium

$$\Gamma = \Gamma_n\left(\mu,\delta\right) + \frac{\mu^2}{\pi^2}(2\delta-\Delta_0)^2 > \Gamma_n\left(\mu,\delta\right)$$

$$\left(\frac{\partial^2\Gamma}{\partial\Delta^2}\right)_{\mu_B,\mu_Q} < 0$$

Ustable!

Stabilization by the charge neutrality ***Shovkovy & Huang***



$$n_Q = -\left(\frac{\partial \Gamma}{\partial \mu_Q}\right)_{\mu,\Delta} = 0$$

---- Along the red line

$$\left(\frac{\partial^2 W}{\partial \Delta^2}\right)_{\mu_B, n_Q} = \left(\frac{\partial^2 \Gamma}{\partial \Delta^2}\right)_{\mu_B, \mu_Q} + \frac{\left(\frac{\partial n_Q}{\partial \Delta}\right)^2_{\mu_B, \mu_Q}}{\left(\frac{\partial n_Q}{\partial \mu_Q}\right)_{\mu_B, \Delta}} > 0$$

$$\Gamma_G < \Gamma_F$$

where $W \equiv \Gamma - \mu_Q n_Q = \Gamma$

---- Generalization to 3 flavors ---> gCFL ***Alford, Bowers & Rajagopal***

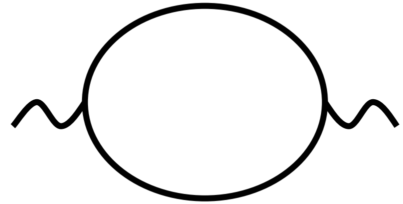
---- Good candidate, but …

Chromomagnetic instability

Huang & Shovkovy

Color Magnetic field $\mathbf{A}^l, \quad l = 1, 2, ..., 8, \quad \nabla \cdot \mathbf{A}^l = 0$

Polarization function $\Pi_{ij}^{l'l}(\mathbf{k}, k_0) =$



$(\mathbf{k}, k_0)$

Inverse penetration depth square $$\frac{1}{d_l^2} = \lim_{\mathbf{k}\to 0}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\Pi_{ij}^{ll}(\mathbf{k}, 0)$$

For (u,d) or (u,d) pairing

$$\frac{1}{d_{1,2,3}^2} = 0$$

$$\frac{1}{d_{4,5,6,7}^2} = \frac{g^2\mu^2}{\pi^2}\left[\frac{\Delta^2 - 2\delta^2}{2\Delta^2} - \frac{\delta\sqrt{\delta^2 - \Delta^2}}{\Delta^2}\theta(\delta - \Delta)\right]$$

$$\frac{1}{d_8^2} = \frac{\left(3g^2 + e^2\right)\mu^2}{27\pi^2}\left[1 - \frac{\delta}{\sqrt{\delta^2 - \Delta^2}}\theta\left(\delta - \Delta\right)\right]$$

Note that $d_{4,5,6,7}^2 < 0 \text{ and } d_8^2 < 0 \text{ for g2SC } (\delta > \Delta)$!

- *2SC-LOFF* (one way out of the instability)

$$<\psi\psi> \propto \Delta e^{2i\mathbf{q}\cdot\mathbf{r}}$$

Alford, Bowers & Rajagopal; Giannakis & Ren

---- Thermodynamic potential:

$$\Gamma \cong \Gamma_n(\mu,\delta) + \frac{2\mu^2}{\pi^2}\left[\Delta^2\left(\ln\frac{\Delta}{\Delta_0} - \frac{1}{2}\right) + \frac{(q+\delta)^3}{4q} f(x_+) + \frac{(q-\delta)^3}{4q} f(x_-)\right]$$

where $$f(x) = (1-x^2)\ln\frac{1+x}{1-x} + \frac{2}{3}(2x^3 - 3x + 1)$$

$$x_\pm = \sqrt{1 - \frac{\Delta^2}{(q\pm\delta)^2}}\,\theta\left(1 - \frac{\Delta}{|q\pm\delta|}\right)$$

---- Equilibrium condition: $\left(\frac{\partial\Gamma}{\partial\Delta}\right)_{\mu,\delta,q} = \left(\frac{\partial\Gamma}{\partial q}\right)_{\mu,\delta,\Delta} = 0$

---- Excitation spectrum: gapless

---- Within the LOFF window, the matrix

$$\begin{pmatrix} \frac{\partial^2\Gamma}{\partial\Delta^2} & \frac{\partial^2\Gamma}{\partial\Delta\partial q} \\ \frac{\partial^2\Gamma}{\partial\Delta\partial q} & \frac{\partial^2\Gamma}{\partial q^2} \end{pmatrix}$$

is positive definite at equilibrium.

---- The inverse penetration depth square: ***Giannakis & Ren***

$$\left(\frac{1}{d_{1,2,3}^2}\right)_{ij} = 0$$

$$\left(\frac{1}{d_{4,5,6,7}^2}\right)_{ij} = \frac{g^2\mu^2}{6\pi^2}\left[A\left(\frac{\Delta}{\delta},\frac{q}{\delta}\right)\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) + B\left(\frac{\Delta}{\delta},\frac{q}{\delta}\right)\frac{q_i q_j}{q^2}\right]$$

$$\left(\frac{1}{d_8^2}\right)_{ij} = \frac{\left(3g^2 + e^2\right)\mu^2}{27\pi^2}\left[C\left(\frac{\Delta}{\delta},\frac{q}{\delta}\right)\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) + D\left(\frac{\Delta}{\delta},\frac{q}{\delta}\right)\frac{q_i q_j}{q^2}\right]$$

The functions A, B, C, D are all positive within the LOFF window
No chromomagnetic instability there.

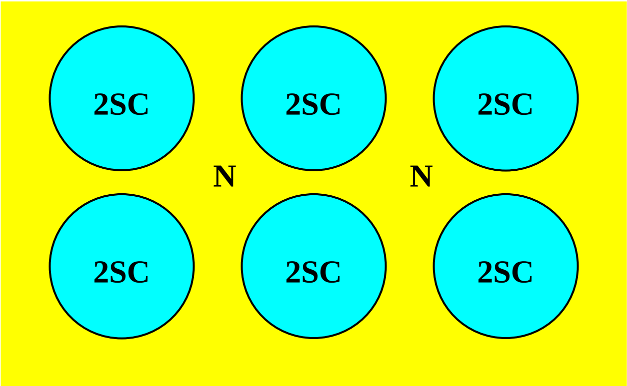
---- A charge neutral LOFF state can be constructed within the
LOFF window. ***Giannakis, Hou & Ren***

---- Generalization to 3 flavors by ***Casalbuoni et. al.***

- ***Other Candidates***

i) Gluon condensation: ***Gorbar, Hashimoto & Miransky***
The condensation induces additional contribution to the inverse penetration depth square through the non Abelian gluon coupling.

ii) Heterotic structure of 2SC and normal phases:
---- similar to the phase separation of cold atom system;
---- microscopic charge fluctuation, macroscopic neutral.



Bedaque, Caldas & Rupak

iii) Equal flavor pairing: ***Schaefer***
---- (u,u) pair or (d,d) pair;
---- non s-wave pairing, beyond NJL model;
---- one-gluon-exchange gives

$$T_c(\text{p - wave}) \cong e^{-\frac{9}{2}} T_c(\text{s - wave})$$

Brown, Liu & Ren; Schaefer

- *Observing CSC in a Neutron Star*

Alford, Bowers & Rajagopal

---- If there is a quark matter core inside a neutron star, the impact of its equation of state may show up in the accretion process in a binary system. But the change of the equation of state by the color superconductivity may be too small to be observed with the present technique.

---- Color superconductivity may be detected by the cooling process via neutrino emission. The change of the cooling rate during the history of a neutron star may indicate formations of various type of energy gaps

---- A neutrino burst from a supernova may be expected because of the CSC transition in a newly born neutron star.

---- The pinning of the rotational vortices by the crystalline of LOFF may contribute to the glitch phenomena of a neutron star.

---- R-mode instability may be triggered by the CSC transition because of the suppression of the shear and bulk viscosities by the gap.

IV. Concluding Remarks

- A cold quark matter of sufficiently high baryon density will become color superconducting at sufficiently low temperature. The transition temperature and the energy gap can be calculated with perturbative QCD. The long range nature of the pairing force gives rise to a non BCS scaling.
- At realistic baryon density, the Fermi momentum mismatch may lead to an exotic superconducting ground state of a quark matter. But a consensus has not been reached.
- It is necessary to pin point a number of clear cut prediction to observe the color superconductivity in nature, especially in a neutron star.
- Color superconductivity is a subject that integrates high energy physics, condensed matter physics, nuclear physics and astrophysics.

Review Articles

- D. Bailin and A. Love, *Phys. Rep*. **107**, 325 (1984).
- K. Rajagopal and F. Wilczek, hep-ph/0011333.
- D. K. Hong, Acta Phys. Polon. B32, 1253 (2001).
- M. Alford, Ann. Rev. Nucl. Part. Sci., 51, 131 (2001).
- T. Schaefer, hep-ph/0304281.
- D. Rischke, Prog. Part. Nucl. Phys. 52, 197 (2004).
- R. Casalbuoni and G. Nardulli, Rev. Mod. Phys., 76, 263 (2004).
- M. Ruballa, Phys. Rept. 407, 205 (2005).
- Hai-cang Ren, CCAST lecture, hep-ph/0404074.
- Mei Huang, CCAST lecture, Int. J. Mod. Phys. E14, 675 (2005).
- I. Shovkovy, Found. Phys. 35, 1309 (2005).
- M. Alford, K. Rajagopal, T. Schaefer and A. Schmitt, arXiv:0709.4635.

Thank you!