Quantum technology, group theory, phase space

Lecture 3, Peking University 2019

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  - Special thanks to my co-authors
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OUTLINE

- Why do we want a phase-space for fermions?
- Gaussian operator bases
- Definitions of Q-functions and P-functions
- Results for completeness and resolution of unity
- Results for differential identities, Fermi shockwaves
- APPLICATIONS TO FOUNDATIONS OF QM MEASUREMENT

Stochastic phase-space methods have many applications for carrying out quantum simulations. Some previous results:

Exact solutions for quantum optical time crystals.

Dynamical simulations of million-mode interacting BECs.

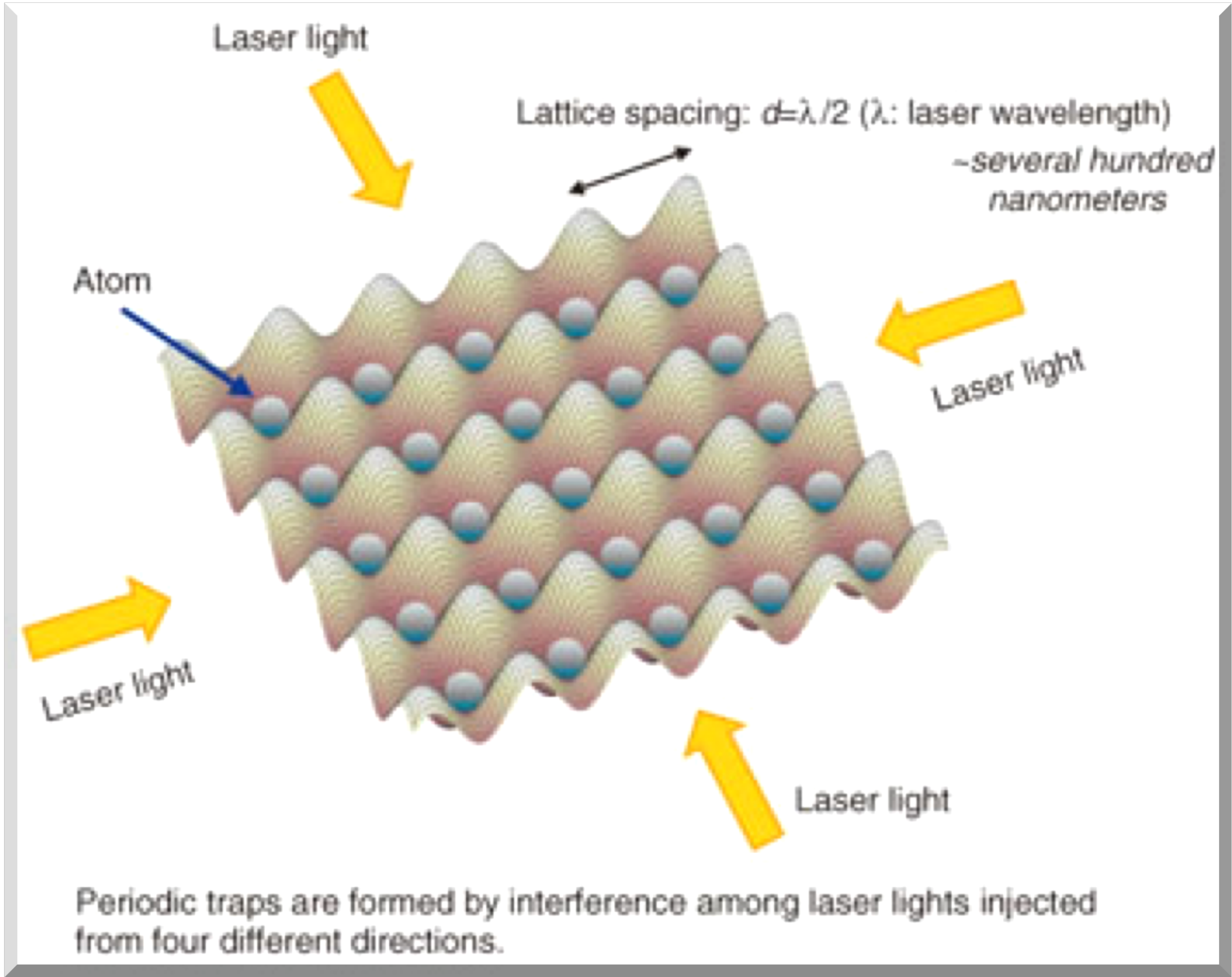
Simulations of optomechanical entanglement.

Fermionic problems are even more challenging: there is usually nc means to sample the density matrix probabilistically. In this talk some recent advances for **fermions** will be treated, including

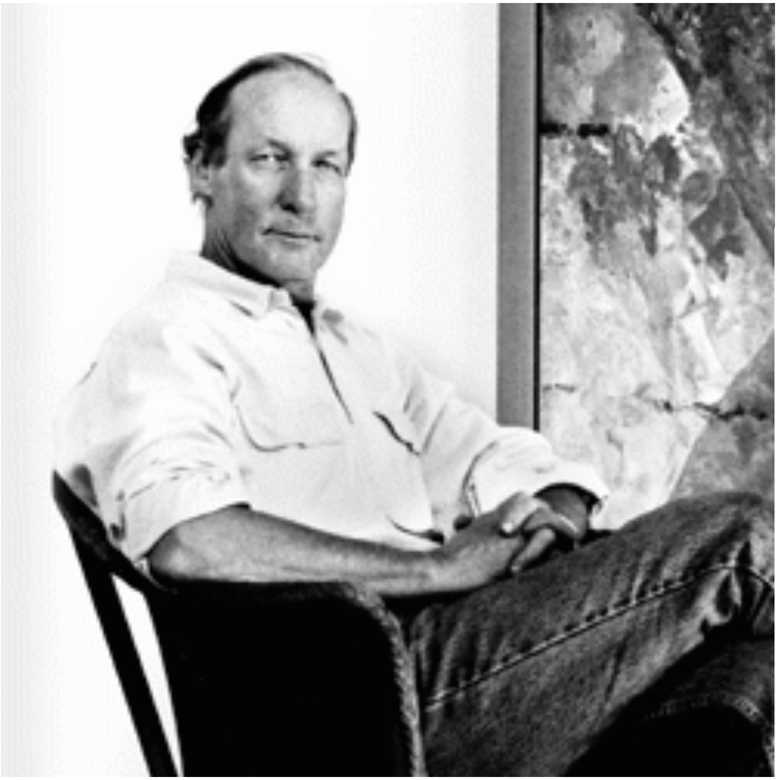
- A generalized Q and P-function definition applied to fermions
- Identities for mapping fermionic operators
- Examples: finite temperature shock waves, collective modes

What about many-body fermion and majorana systems?

Using trapped fermions to model High Tc superconductors: Fermi-Hubbard model



Fermi-Hubbard model



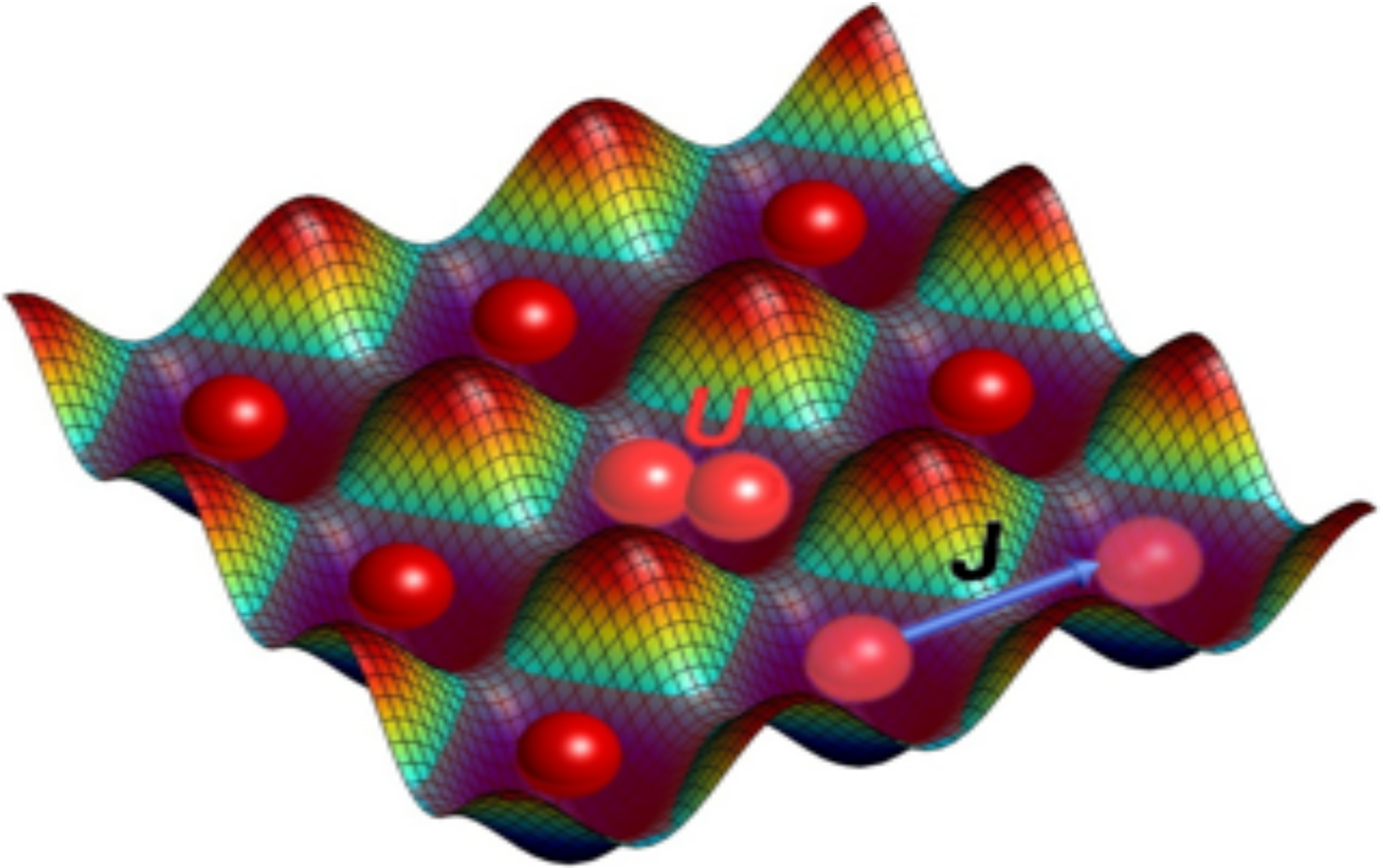
$$\widehat{H} = \sum_{i,j,\sigma}^{M} t_{ij}\widehat{c}_{i\sigma}^{\dagger}\widehat{c}_{j\sigma} + U\sum_{i}^{M}\widehat{c}_{i\uparrow}^{\dagger}\widehat{c}_{i\uparrow}\widehat{c}_{i\downarrow}^{\dagger}\widehat{c}_{i\downarrow}$$

Where:

- U as the on-site Coulomb interaction,
- μ the chemical potential,
- M the number of the lattice sites and
- the transfer integral between the i-th site and the j-th site as

$$t_{ij} = \begin{cases} -\mu & \text{for} \quad i = j \\ -t & \text{for} \quad (i,j) \quad \text{being a nearest neighbour} \\ 0 & \text{otherwise} \end{cases}$$

Simple picture: Problem – the ground state is still unknown!



Exponential complexity problem

- Experiments (Harvard) are at 0.25 of the Fermi temperature
- Many different geometries and dynamical properties accessible
- Many-body Hilbert space is exponentially complex
- No reliable method for calculating long-range order.
- How can we understand complex Fermi systems - sign problems?

- **Is there another approach apart from number states?**

J. F. Corney and P. D. D., Phys. Rev. B **73**, 125112 (2006). (Queensland)
T. Aimi and M. Imada, J. Phys. Soc. Jpn., **76**, 084709 (2007). (Tokyo Uni)
A. Mazurenko, et. al., Nature 545, 462-466 (2017). (Harvard Experiment, Greiner)

Suppose we have a positive definite, hermitian operator basis $\hat{\Lambda}\left(\vec{\lambda}\right)$ defined in a Hilbert space $\mathcal{H}$ of quantum mechanical operators, where $\vec{\lambda}$ is a vector in the phase-space domain $\mathcal{D}$.

- A generalized Q-function is defined as the inner product of the density matrix $\hat{\rho}$ with the operator basis:

$$Q\left(\vec{\lambda}\right) = \mathrm{Tr}\left[\hat{\Lambda}\left(\vec{\lambda}\right)\hat{\rho}\right].$$

- Physical interpretation: this is simply the probability of observing the system in the state $\hat{\Lambda}\left(\vec{\lambda}\right)$.

K. Husimi, Proc. Phys. Math, Soc. Jpn. 22, 264 (1940)

Suppose we have an operator basis $\hat{\Lambda}\left(\vec{\lambda}\right)$ defined in a Hilbert space $\mathcal{H}$ of quantum mechanical operators, where $\vec{\lambda}$ is a vector in the phase-space domain $\mathcal{D}$.

- A generalized P-function is defined as an expansion of the density matrix $\hat{\rho}$ using the operator basis:

$$\hat{\rho} = \int_{\mathcal{D}} P\left(\vec{\lambda}\right) \hat{\Lambda}\left(\vec{\lambda}\right) d\mu\left(\vec{\lambda}\right) .$$

- Physical interpretation: this is simply the expansion of the density matrix in terms of states $\hat{\Lambda}\left(\vec{\lambda}\right)$.
- This does not require hermiticity or positivity of $\hat{\Lambda}\left(\vec{\lambda}\right)$, and
- Generally different to Q owing to non-orthogonality; can use a different domain.

Roy J. Glauber, Phys. Rev. 131, 2766 (1963).
P. D. D. and C. W. Gardiner, J. Phys. **A13**, 2353-2368 (1980).

Completeness

We require the following completeness property: the identity operator $\hat{I}$ of the Hilbert space can be resolved as an integral over the phase-space, so that

$$\int_{\mathcal{D}} \hat{\Lambda}\left(\vec{\lambda}\right) d\mu\left(\vec{\lambda}\right) = \hat{I}\,.$$

This is called a resolution of unity.
Here $d\mu\left(\vec{\lambda}\right)$ is an associated integration measure on the phase-space.

- A set of identities that allows all operator moments of physical interest $\hat{O}_n$ to be mapped into differential operators is required, so that:

$$\hat{O}_n \hat{\Lambda}\left(\vec{\lambda}\right) = \mathcal{D}_n\left(\partial_{\vec{\lambda}}, \vec{\lambda}\right)\hat{\Lambda}\left(\vec{\lambda}\right)$$

Using the resolution of unity, any observable in the form of an operator moment can be represented as:

$$\left\langle \hat{O}_n \right\rangle = \int \mathcal{D}_n\left(\partial_{\vec{\lambda}}, \vec{\lambda}\right) Q\left(\vec{\lambda}\right) d\vec{\lambda}.$$

Fermion case

Here we consider normally-ordered Gaussian operators, with unit trace:

$$\hat{\Lambda}\left(\underline{\underline{\sigma}}\right) = \sqrt{\det\left[i\underline{\underline{\sigma}}\right]} : \exp\left[-\underline{\hat{a}}^{\dagger}\left(\underline{\underline{\sigma}}^{-1} - 2\underline{\underline{\bar{I}}}\right)\underline{\hat{a}}/2\right] : ,$$

with:

$$\underline{\underline{\bar{I}}} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \text{and} \qquad \underline{\underline{\sigma}} = \begin{bmatrix} \mathbf{n}^T - \mathbf{I} & \mathbf{m} \\ -\mathbf{m}^* & \mathbf{I} - \mathbf{n} \end{bmatrix}.$$

The $2M \times 2M$ matrix $\underline{\underline{\sigma}}$ is the *covariance* matrix expressed in terms of an $M \times M$ hermitian matrix $\mathbf{n}$ and a complex antisymmetric $M \times M$ matrix $\mathbf{m}$.

We define a "stretched" variable $\underline{\underline{\zeta}}$ as:

$$\underline{\underline{\zeta}} = \underline{\underline{\bar{I}}} - 2\underline{\underline{\sigma}} = \underline{\underline{\tilde{\sigma}}} - \underline{\underline{\sigma}}.$$

We also define a normalized Gaussian basis $\hat{\Lambda}^N$, which in terms of these variables is:

$$\hat{\Lambda}^N\left(\underline{\underline{\zeta}}\right) = \frac{1}{\mathcal{N}}\hat{\Lambda}\left(\frac{1}{2}\left[\underline{\underline{\bar{I}}} - \underline{\underline{\zeta}}\right]\right) S\left(\underline{\underline{\zeta}}^2\right).$$

$S\left(\underline{\underline{\zeta}}^2\right)$ is an even, positive scaling function, invariant under unitary transformations. These operators have the class-D symmetry introduced by Altland and Zirnbauer.

A. Altland and M. R. Zirnbauer. Phys. Rev. B **55** 1142 (1997)

Resolution of Unity: use matrix polar coordinates (L.K. Hua)

The Gaussian operator $\hat{\Lambda}^N\left(\underline{\underline{\zeta}}\right)$ are the basis for the fermionic Q-function. This is a positive hermitian basis.

We have proved the following resolution of unity:

$$\hat{I} = \int_{\mathcal{D}} d\zeta \hat{\Lambda}^N\left(\underline{\underline{\zeta}}\right),$$

where $d\zeta$ is the Riemannian measure on the symmetric space.

L. K. Hua, Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains (1963).
Laura E. C. Rosales-Zarate, P. D. D., Journal of Physics A **46**, 275203 (2013).

The fermionic Q-function is defined for any density matrix, in terms of the Gaussian basis, as:

$$Q\left(\underline{\underline{\zeta}}\right) = \mathrm{Tr}\left[\hat{\rho}\hat{\Lambda}^{N}\left(\underline{\underline{\zeta}}\right)\right].$$

- The Gaussian operators and the density matrix are positive definite hence the Q-function is positive.
- From the resolution of unity, the Q-function is normalized to unity: $\int d\zeta Q\left(\underline{\underline{\zeta}}\right) = 1$.
- This fermionic Q-function is a non-negative probability distribution, and it is normalized to unity. It is also defined for any density-matrix.

In the extended variables, observables can be expressed in normal, antinormal or nested ordering. We consider the antinormal form of the observables, which are given by:

$$\left\langle \left\{ \underline{\hat{a}\hat{a}^{\dagger}} \right\} \right\rangle = \mathrm{Tr}\left[\hat{\rho} \begin{pmatrix} \hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^{\dagger} & \hat{\boldsymbol{a}}\hat{\boldsymbol{a}}^{T} \\ \hat{\boldsymbol{a}}^{\dagger T}\hat{\boldsymbol{a}}^{\dagger} & -\left(\hat{\boldsymbol{a}}^{\dagger T}\hat{\boldsymbol{a}}^{T}\right)^{T} \end{pmatrix} \right].$$

Using the resolution of unity, the observables can be expressed as:

$$\left\langle \left\{ \hat{a}_{\alpha}\hat{a}_{\beta}^{\dagger} \right\} \right\rangle = \mathrm{Tr}\left[\int d\zeta \hat{\rho} \left\{ \hat{a}_{\alpha} : \hat{a}_{\beta}^{\dagger}\hat{\Lambda}^{N} : \right\} \right].$$

We prove the following differential identity

$$\left\{\underline{\hat{a}} : \underline{\hat{a}}^{\dagger}\hat{\Lambda}^{N} :\right\} = -\underline{\underline{\sigma}}\hat{\Lambda}^{N} - \underline{\underline{\tilde{\sigma}}}\frac{\partial\hat{\Lambda}^{N}}{\partial\underline{\underline{\sigma}}}\underline{\underline{\sigma}} + \underline{\underline{\tilde{\sigma}}}\hat{\Lambda}^{N}\frac{\partial\ln S}{\partial\underline{\underline{\sigma}}}\underline{\underline{\sigma}}.$$

Considering the explicit form for the normalization function S, in the limit $S \to 1$, the observables are given by:

$$\begin{aligned} \left\langle\left\{\underline{\hat{a}\hat{a}^{\dagger}}\right\}\right\rangle &= C_M \int_{\mathcal{V}} \underline{\underline{\zeta}} Q\left(\underline{\underline{\zeta}}\right) d\zeta - \frac{1}{2}\underline{\underline{\bar{I}}}, \\ C_M &= 2M - 1/2. \end{aligned}$$

Laura Rosales-Zarate and P. D. D., New J. Phys. **17**, 032002 (2015).

Thermal Q-function

The density matrix of a thermal state is:

$\hat{\rho}_{th} = \tilde{n}_{th} : \exp\left[-\hat{a}^{+}\left(2 - \tilde{n}_{th}^{-1}\right)\hat{a}\right] := \hat{\Lambda}_{th}\left(n_{th}\right)$

- In this case the Q-function is:

$$\begin{aligned} Q_{th}(\zeta) &= \mathrm{Tr}\left[\hat{\Lambda}_{\mathrm{th}}\left(\mathrm{n}_{\mathrm{th}}\right)\ \hat{\Lambda}_{1}^{\mathrm{N}}(\zeta)\right] = \frac{1}{2\mathcal{N}}\mathrm{S}_{1}\left(\zeta^{2}\right)\left(1+\zeta_{\mathrm{th}}\zeta\right),. \\ S_{1}\left(\zeta^{2}\right) &= \left(1-\zeta^{2}\right)^{k} \end{aligned}$$

- Observables are given by:

$$\left\langle 2\hat{a}\hat{a}^{\dagger} - 1\right\rangle_{th} = 3\int_{-1}^{1}\zeta Q_{th}\left(\zeta\right)d\zeta = \zeta_{th}$$

Next we consider Majorana operators, with:

$$\gamma_1 = a + a^\dagger$$
$$\gamma_2 = i\left(a^\dagger - a\right)$$

unit trace Gaussian opertor:

$$\hat{\Lambda}\left(\underline{\underline{x}}\right) = N\left(\underline{\underline{x}}\right) : \exp\left[-i\underline{\hat{\gamma}}^T \underline{\underline{\mathcal{I}}}\left[\underline{\underline{I}} - \left(\underline{\underline{\mathcal{I}}}\underline{\underline{x}} + \underline{\underline{I}}\right)^{-1}\right]\underline{\hat{\gamma}}/2\right] : .$$

Here $N\left(\underline{\underline{x}}\right) = \frac{1}{2^M}\sqrt{\det\left[\underline{\underline{\mathcal{I}}} - \underline{\underline{x}}\right]}$, $\underline{\underline{\mathcal{I}}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$ and $\underline{\underline{I}}$ is the $2M \times 2M$ identity matrix.

$$\langle O \rangle \;=\; \int_{\mathcal{D}_C} P\left(\underline{\underline{x}}, \tau\right) \mathrm{Tr}\left[O \hat{\Lambda}\left(\underline{\underline{x}}\right)\right] d\underline{\underline{x}} \equiv \left\langle O\left(\underline{\underline{x}}\right)\right\rangle_P .$$

We use the unordered differential identities derived from those given previously to obtain the observable function $O\left(\underline{\underline{x}}\right)$. We consider the correlation function $\hat{X}_{\mu\nu}$ given by:

$$\hat{X}_{\mu\nu} \equiv \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right].$$

$$\left\langle \underline{\underline{\widehat{X}}} \right\rangle = \int \underline{\underline{x}} P\left(\underline{\underline{x}}\right) d\underline{\underline{x}},$$

R. R. Joseph, L. E. C. Rosales-Zárate, P. D. D., Journal of Physics A: Mathematical and Theoretical **51**, 245302 (2018).
R. R. Joseph, L. E. C. Rosales-Zárate, P. D. D., Physical Review A **98**, 013638 (2018).

Classical domains of E. Cartan

1. The domain $\mathcal{R}_I$ of $m \times n$ complex matrices with: $\underline{\underline{I}}^{(m)} - \underline{\underline{ZZ}}^{\dagger} > 0.$
2. The domain $\mathcal{R}_{II}$ of $n \times n$ symmetric complex matrices with: $\underline{\underline{I}} - \underline{\underline{ZZ}}^{*} > 0.$
3. The domain $\mathcal{R}_{III}$ of $n \times n$ skew-symmetric (anti-symmetric) complex matrices with: $\underline{\underline{I}} + \underline{\underline{ZZ}}^{*} > 0.$
4. The domain $\mathcal{R}_{IV}$ of n-dimensional vectors $z = (z_1, z_2, \ldots, z_n)$, where z_k are complex numbers, satisfying: $\left|zz^T\right| + 1 - 2zz^T > 0, \qquad \left|zz^T\right| < 1.$

M-phase space

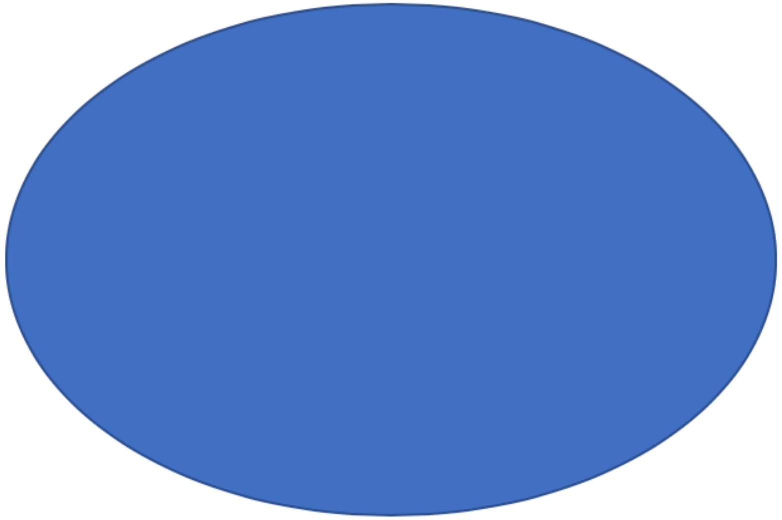
This is a REAL antisymmetric phase space:

$$\underline{\underline{x}} = \begin{bmatrix} i\left(\mathbf{n}^- + \mathbf{m}^-\right) & \mathbf{n}^+ + \mathbf{m}^+ - \mathbf{I} \\ -\mathbf{n}^+ + \mathbf{m}^+ + \mathbf{I} & i\left(\mathbf{n}^- - \mathbf{m}^-\right) \end{bmatrix}$$

where $\mathbf{n}^\pm = \mathbf{n} \pm \mathbf{n}^T$, $\mathbf{m}^\pm = \mathbf{m} \pm \mathbf{m}^*$ and $n_{ij} = \left\langle \hat{a}_i^\dagger \hat{a}_j \right\rangle$, $m_{ij} = \langle \hat{a}_i \hat{a}_j \rangle$.

$$\underline{\underline{x}}\,\underline{\underline{x}}^T - \underline{\underline{I}} < 0$$

Bounded real domain in
M(2M-1) = 1,6,15,28,..   dimensions:



Hamiltonian

$$\widehat{H} = \hbar\omega_{ij}\widehat{a}_i^\dagger\widehat{a}_j,$$

If we define the Majorana commutator as previously

$$\underline{\underline{\Omega}} = \begin{bmatrix} 0 & \boldsymbol{\omega} \\ -\boldsymbol{\omega} & 0 \end{bmatrix},$$

we can re-express the Hamiltonian in terms of Majorana operators as

$$\hat{H} = \frac{\hbar}{2}\Omega_{\mu\nu}\hat{X}_{\mu\nu}.$$

Time-evolution

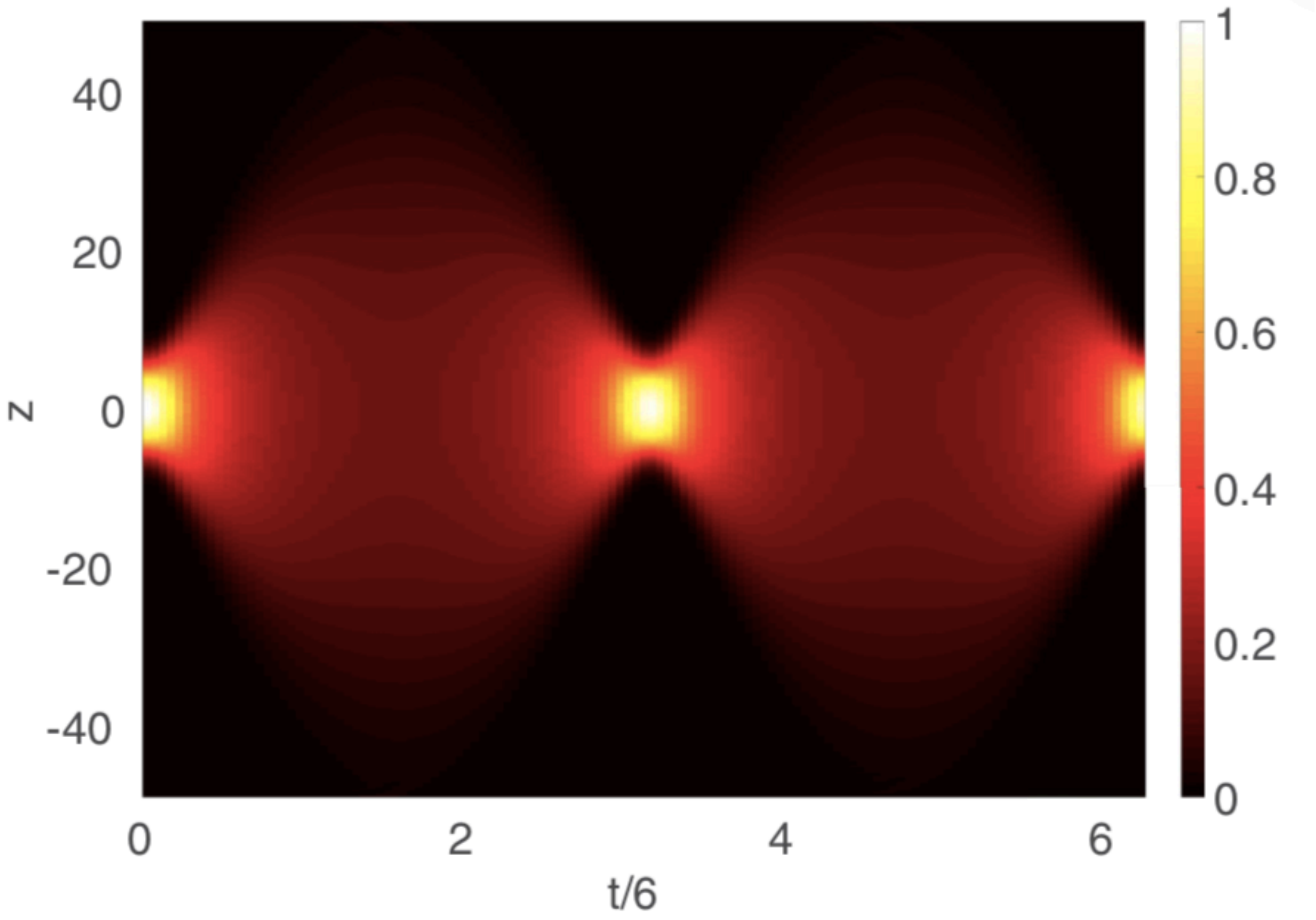
$$\frac{dQ\left(\underline{\underline{x}}\right)}{dt} = \Omega_{\mu\nu}\left[\frac{d}{dx_{\nu\kappa}}\left(x_{\mu\kappa}Q\right) - \frac{d}{dx_{\kappa\mu}}\left(x_{\kappa\nu}Q\right)\right].$$

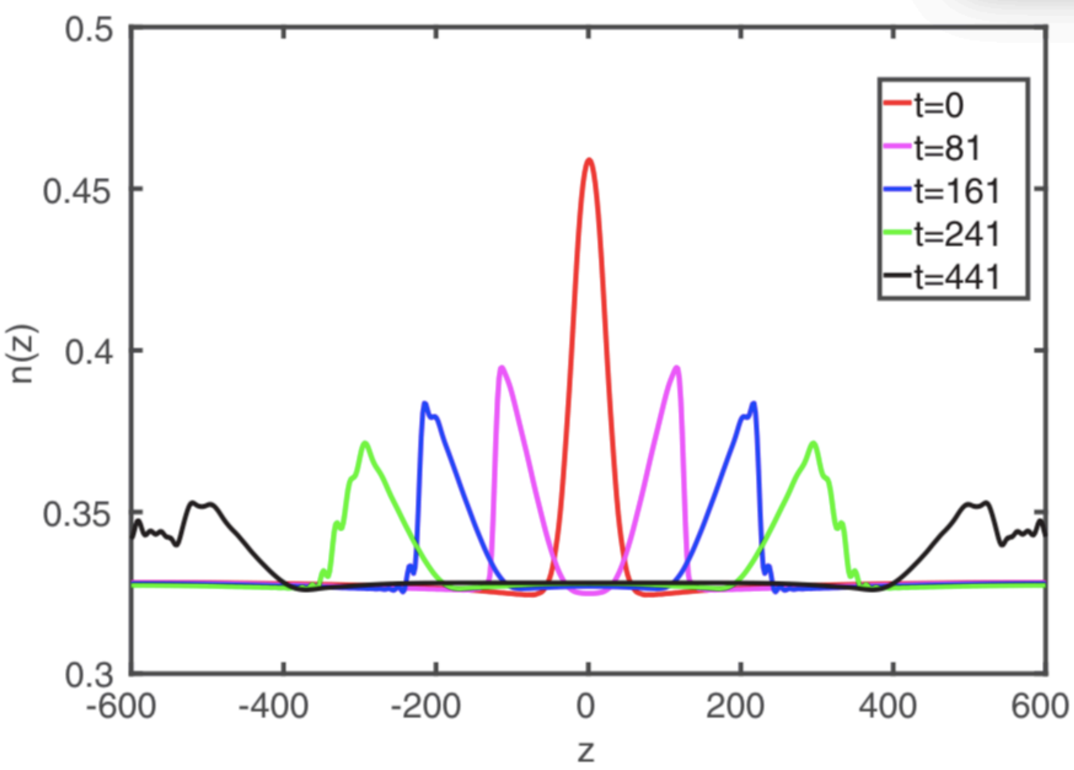
This leads to a characteristic equation for all stochastic trajectories (P or Q)

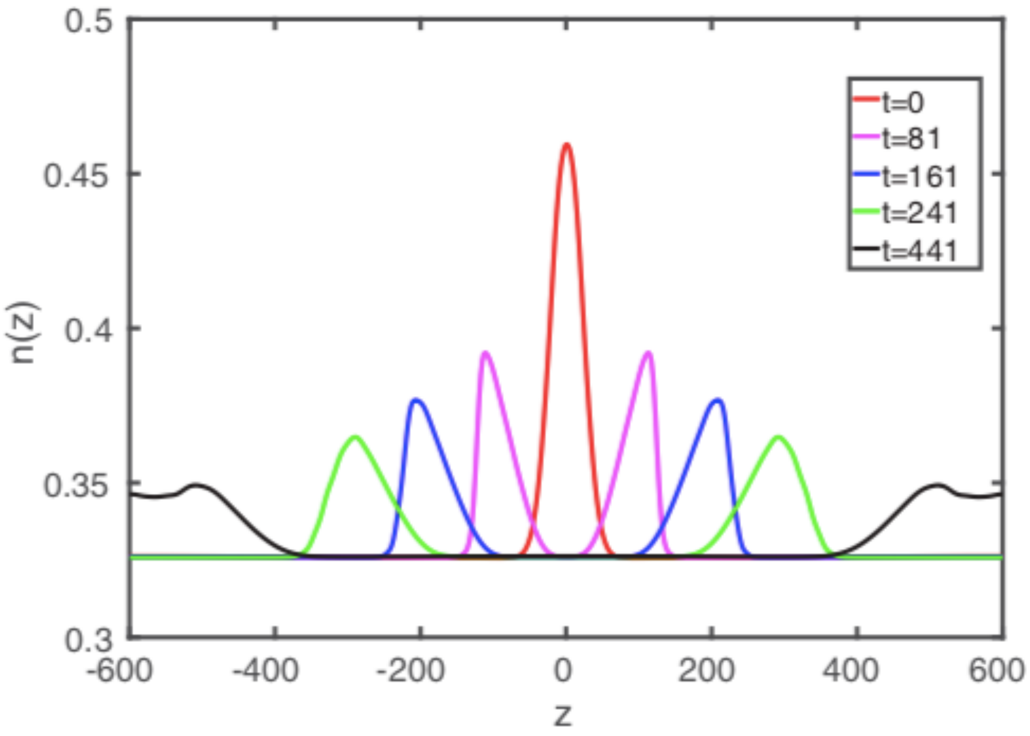
$$\frac{d\underline{\underline{x}}}{dt} = \left[\underline{\underline{\Omega}}, \underline{\underline{x}}\right].$$

Finite temperature breathing oscillations

- Oscillations triggered by a sudden reduction In trap frequency; non-interacting Fermi gas in 1D







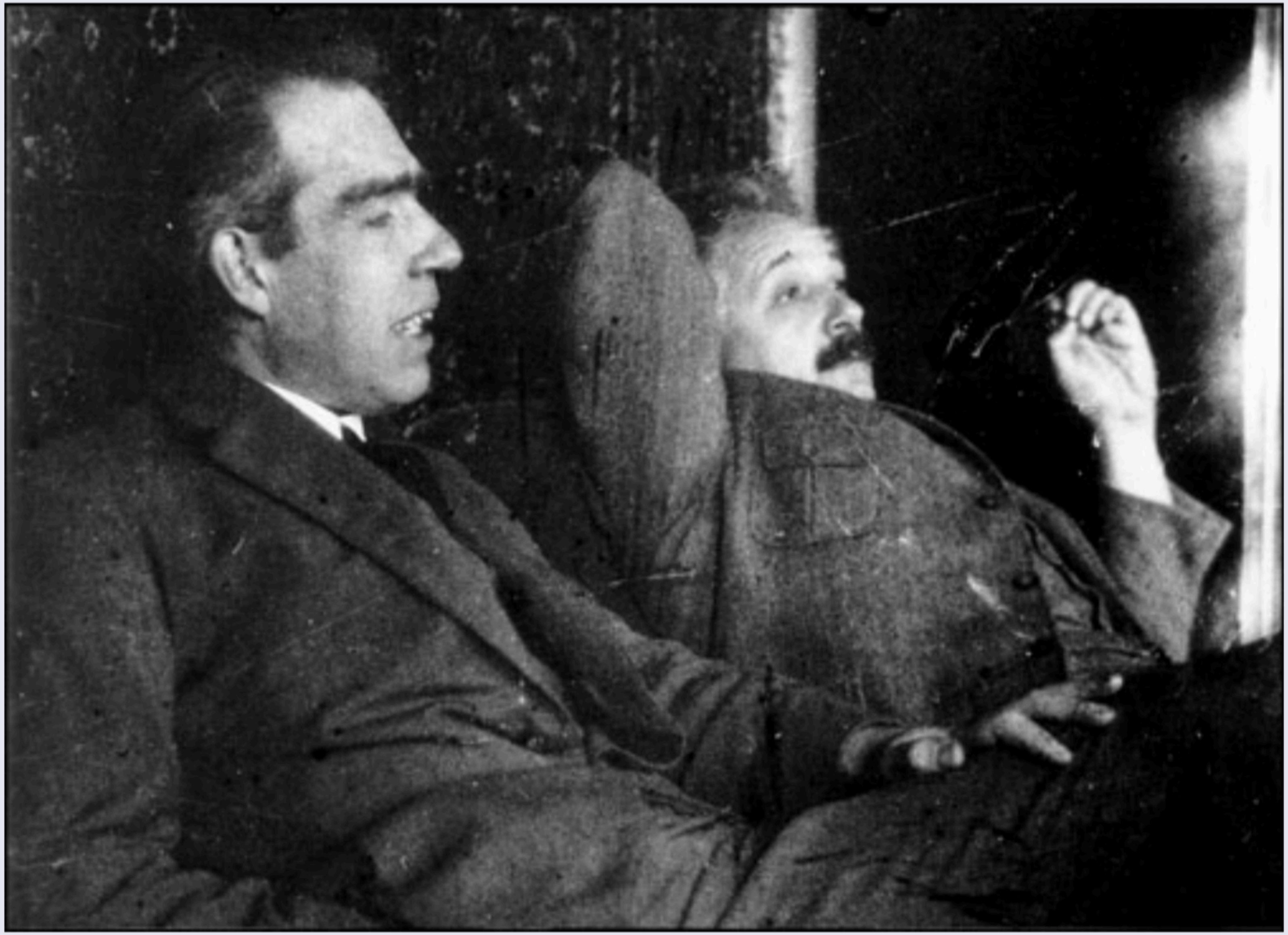
Zero and finite temperature shock waves

Problems with quantum measurement

Measurement is regarded as a non-unitary projection

- But how is a measurement defined?
- Why is it different to unitary physics?
- What if the observer is part of the universe?
- How do we treat quantum cosmology?
- Problem is not solved through decoherence
- This does not project one eigenvalue

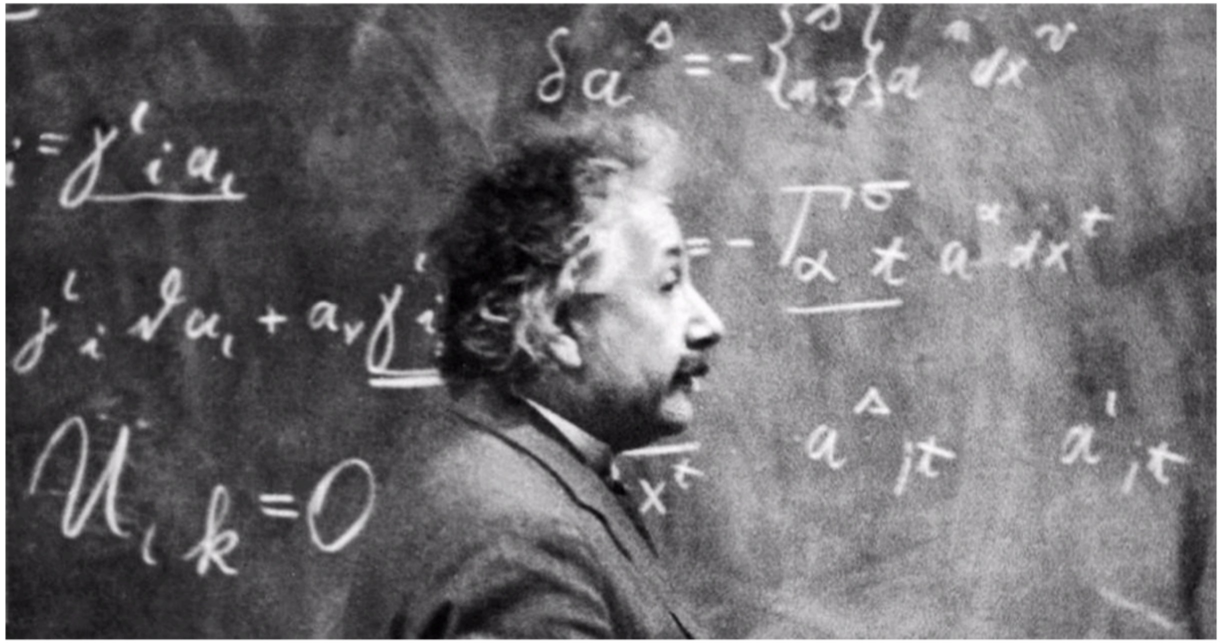
Quantum measurement discussed by Einstein, Bohr



Einstein and Bohr, at the Solvay conferences, 1927-1930.

Screenshot

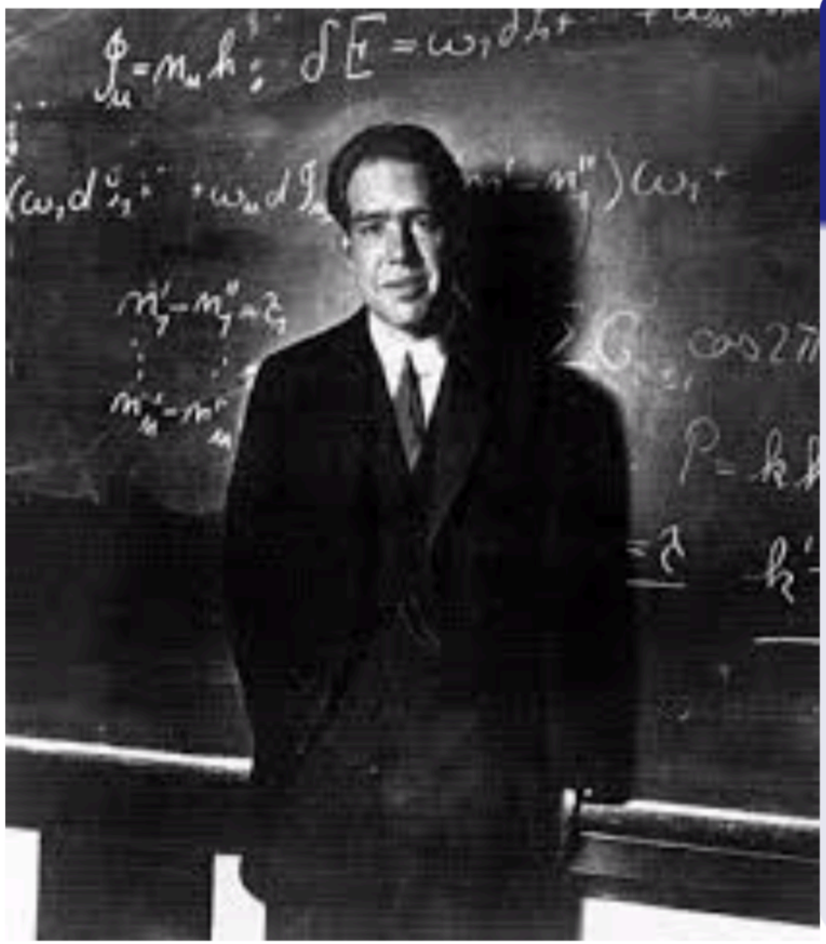
What did Einstein believe?



Einstein: a physical theory must be

- **Objective**: in the sense of observer independent
- **Local**: in terms of fields in space and time
- **Complete**: all that exists should be included

What about Bohr?



Bohr's ideas about quantum theory

- **Operational** measurement must be included
- **Wave functions** are symbolic, not real
- **Contextuality** is fundamental

Screenshot

These goals are not in contradiction!

Ideally a physical theory should be

- **Objective:** external entities exist
- **Measurable:** through physical operations
- **Local:** objects are localized in spacetime
- **Consistent:** no special treatment of measurement
- **Stochastic:** we only have partial knowledge
- **Unified**: observers are part of the universe

Q-functions

The Q-function proves objective models exist

$$Q(\boldsymbol{\lambda},t) = Tr\left\{\hat{\Lambda}(\boldsymbol{\lambda})\hat{\rho}\left(t\right)\right\},$$

- where $\hat{\rho}\left(t\right)$ is the quantum density matrix,
- $\hat{\Lambda}(\lambda) = \prod_{b,f}\hat{\Lambda}_b(\boldsymbol{\psi})\hat{\Lambda}_f\left(\boldsymbol{x}\right)$ is a positive-definite operator basis,
- $\boldsymbol{\lambda}$ is a point in the phase-space.

This must give an expansion of the Hilbert space identity operator $\hat{I}$, so that, given an integration measure $d\boldsymbol{\lambda}$,

- $\hat{I} = \int \hat{\Lambda}(\boldsymbol{\lambda})\,d\boldsymbol{\lambda}$.

Quantum fields

$N-$component bosonic field $\hat{\boldsymbol{\psi}}(r)$

- Defined with a space-time coordinate r, where $r = \left(r^1, \ldots r^{n_d}\right) = (\boldsymbol{r}, t)$.
- Quantum fields $\hat{\psi}_i(\boldsymbol{r})$ are expanded using M operators $\hat{a}_i, \hat{a}_i^\dagger$ for M/N modes.
- The indices i include
- N internal degrees of freedom

Projection operators

Bosonic case

For bosonic fields, $\hat{\Lambda}$ is proportional to a coherent state projector,

$$\hat{\Lambda}(\boldsymbol{\alpha}) \equiv |\boldsymbol{\alpha}\rangle_c \langle\boldsymbol{\alpha}|_c / \pi^M.$$

The state $|\boldsymbol{\alpha}\rangle_c$ is a normalized Bargmann-Glauber coherent state with $\hat{a}_i|\boldsymbol{\alpha}\rangle_c = \alpha_i|\boldsymbol{\alpha}\rangle_c$ and

$$\hat{\boldsymbol{\psi}}(x)|\boldsymbol{\alpha}\rangle_c = \boldsymbol{\psi}(x)|\boldsymbol{\alpha}\rangle_c,$$

Hence, phase-space is composed of local fields in space-time.

$$\lambda \to \boldsymbol{\psi}(x)$$

This is a probabilistic representation

- $\int Q(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} = 1$
- $Q(\boldsymbol{\lambda}) \geq 0$

Every phase-space coordinate is a possible universe

- Each $\lambda(t)$ is a set of fields in space time
- Every $\lambda(t)$ is a possible universe
- There is only one objective universe

Observables

We can compute observables in the usual way

Quantum expectations $\left\langle \hat{O} \right\rangle_Q$ of ordered observables $\hat{O}$ are identical to classical probabilistic averages $\langle O \rangle_C$ - including corrections for operator re-ordering if necessary - so that:

$$\left\langle \hat{O} \right\rangle_Q = \langle O \rangle_C \equiv \int d\boldsymbol{\lambda} \, Q(\boldsymbol{\lambda}) \, O(\boldsymbol{\lambda}) \, .$$

We can also add a model of the measurement, which models the growth of the observable to a macroscopic size.

Differential equations from operator mappings

Operator correspondences→Fokker-Planck equation

Fokker-Planck equation is of the form:

$$\dot{Q} = \left[-\partial_\mu A_\mu + \frac{1}{2} \partial_\mu \partial_\nu D_{\mu\nu} \right] Q$$

However, D is not positive definite, and in fact $Tr(D) = 0$, yet Q *is* positive definite. This is because it corresponds to a simultaneous positive and negative time evolution. This has boundary conditions in the past and the future, and has a real action principle.

Amplification

Amplification is fundamental to measurement!

From the amplified macroscopic value X, with gain G the experimentalist *infers* an eigenvalue of $\tilde{X} = X_0 + \varepsilon/G$, with a probability distribution of
$P\left(\tilde{X}\right) = \left(G/\sqrt{2\pi}\right) exp\left(-G^2\left(\tilde{X} - X_0\right)/2\right)$

- Vacuum fluctuations relatively negligible at large gain, allowing eigenvalue measurement.

Conclusions

Q function as a fundamental quantum theory

- **Can obtain exact quantum equations**
- Probabilistic
- No requirement to collapse wave function
- **Observer can be included**
- Local fields in space and time
- **Unifies ideas of Bohr and Einstein?**

PDD and Margaret Reid, arXiv:1909.01798
PDD, arXiv:1910.00001